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Uniform Rational Weighted Approximations Having Restricted Ranges[1]

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1. INTRODUCTION

The references at the end of this paper include a number of papers which deal with uniform approximation using a generalized weight function. An important application of the general theory is to the problem of obtaining starting values for the Newton–Raphson iterative schemes for calculating inverses of certain functions ([5], [6]). In working in this area it became evident that a theory of uniform approximations with restricted ranges was needed. To be specific, suppose b_1 and b_2 are real-valued continuous functions on an interval I, and $b_2(x) > b_1(x)$ for all $x \in I$. Suppose $f \in C(I)$ is to be approximated. Then the problem is to find a best uniform approximation to f from a certain family R whose members r satisfy $r \in C(I)$ and $b_1(x) \leqslant r(x) \leqslant b_2(x)$ for all $x \in I$. This restricted range problem is a special case of the more general problem treated in this paper.

2. STATEMENT OF THE PROBLEM

Let X be a compact topological space. Let u and l be two given elements of $C(X)$ such that $l(x) < u(x)$ for all $x \in X$. Let P be an n ($\geqslant 1$) dimensional linear subspace in $C(X)$, and let Q be an m ($\geqslant 1$) dimensional subspace in $C(X)$. Define

$$R = \{r \equiv p/q : p \in P, q \in Q, q(x) > 0 \ \forall \ x \in X\}. \tag{1}$$

If E represents the real axis, we consider a real-valued function $W(x,y)$, with domain $X \times E$, satisfying the following properties:

If

$$D = \{(x,y) \in X \times E : l(x) \leqslant y \leqslant u(x)\}, \tag{2}$$

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then:

(a) W is continuous over D;

(b) $(x,y),(x,z) \in D$ and $y < z \Rightarrow W(x,y) < W(x,z)$;

(c) $(x,y) \in D \Rightarrow \operatorname{sgn} W(x,y) = \operatorname{sgn} y$;

(d) $x \in X$ and $y > u(x) \Rightarrow W(x,y) = \infty$;

(e) $x \in X$ and $y < l(x) \Rightarrow W(x,y) = -\infty$.

A function W with the above properties is called a generalized weight function. Note that (2c) is intended to mean $W(x,0) = 0$ provided $l(x) \leqslant 0 \leqslant u(x)$.

Let $f \in C(X)$ be a function to be approximated. Then the problem considered in this paper is that of finding an $r \in R$ such that

$$\sup_{x \in X} |W[x, f(x) - r(x)]| = \inf_{\tilde{r} \in R} \sup_{x \in X} |W[x, f(x) - \tilde{r}(x)]|. \tag{3}$$

Throughout this paper we shall use the notation

$$M[f-r] \equiv \sup_{x \in X} |W[x, f(x) - r(x)]|.$$

We refer to (3) as "the problem to be solved," and we assume that the problem is always such that $\inf_{r \in R} M[f-r] < \infty$.

3. Applications

Suppose b_1 and b_2 are functions as in the introduction. Then define

$$u(x) \equiv f(x) - b_1(x)$$

$$l(x) \equiv f(x) - b_2(x)$$

$$W(x,y) = \begin{cases} +\infty & y > u(x) \\ y & l(x) \leqslant y \leqslant u(x) \\ -\infty & y < l(x). \end{cases} \tag{4}$$

For this generalized weight function the problem (3) is the restricted range problem given in the introduction.

The standard one-sided approximation problem can be shown to be of the form (3) in the following way. Define

$$W(x,y) = \begin{cases} +\infty & y > 0 \\ y & y \leqslant 0. \end{cases} \tag{5}$$

The problem (3) for the weight function (5) is that of finding an $r \in R$ such that $f(x) - r(x) \leqslant 0 \ \forall\, x \in X$, and for which

$$\max_{x \in X} |f(x) - r(x) \equiv \|f - r\| = \text{minimum}.$$

Here $u(x) \equiv 0$. An $l(x)$ could be defined as follows. Let r_0 be any element of R such that $f(x) - r_0(x) \leqslant 0 \ \forall \ x \in X$. Define

$$l(x) \equiv -\|f - r_0\| - 1.$$

Then (5) can be written in the form (4) using the above u and l.

The function $W(x, y)$ might be rather complicated. The paper [5] is concerned with finding an approximation to $x^{1/2}$ on $X \equiv [a,b]$, where $0 < a < b$; the approximation is to be used to provide starting values for the standard Newton–Raphson iteration to compute an accurate approximation to $x^{1/2}$. It turns out that the problem can be subsumed by (3). Let $\epsilon > 0$ satisfy $\epsilon \ll a^{1/2}$. Then define

$$W(x, y) = \begin{cases} +\infty & y > x^{1/2} - \epsilon \\ \dfrac{(\operatorname{sgn} y) y^2}{2[x - yx^{1/2}]} & y \leqslant x^{1/2} - \epsilon. \end{cases} \tag{6}$$

Solutions to (3) using the weight function (6) provide optimal starting value functions for the Newton–Raphson iterative calculation of $x^{1/2}$ on $[a,b]$. Notice that (6) is somewhat like (5) and is readily modified to fit the hypotheses of (2).

4. Existence

As is usually the case in rational approximation theory, we are not able to give a universal existence theorem. Therefore, in this section we restrict our attention to the case where X is a real interval and R consists of functions of the form

$$r \equiv \frac{p}{q} \equiv \frac{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}}{b_0 + b_1 x + \cdots + b_{m-1} x^{m-1}}. \tag{7}$$

Here n and m are fixed nonzero positive integers.

PROPOSITION. *Let R be the set of rational functions of the form* (7), *and let X be an interval* $[a,b]$. *If there exists an $r \in R$ for which $M[f - r] < \infty$, then the problem* (3) *has a solution.*

Proof. Let

$$\rho = \inf_{r \in R} M[f - r].$$

Then one can restrict one's attention to those $r \in R$ for which $M[f - r] \leqslant \rho + 1$. Using the properties (2) it follows that there exists a B such that $M[f - r] \leqslant \rho + 1$ implies $\|f - r\| \leqslant B$.

From here on the existence proof does not differ appreciably from that in the unrestricted range case discussed in [7]. Thus, we omit the details.

5. Characterization Theorem

As in ordinary uniform rational approximation theory, the key theorem in the theory of (3) deals with the question whether or not the origin of a certain Euclidean space lies in some particular convex hull. The result for the generalized weight function case (2) is somewhat more involved than for ordinary rational approximation.

Consider an $r \in R$, where R is as in (1), for which $M[f-r] < \infty$. Then define

$$S_r = \{h \equiv p + rq : p \in P, q \in Q\}. \tag{8}$$

The set S_r is a linear space of dimension $s \leqslant n + m - 1$. Let $g_1(x), g_2(x), \ldots, g_s(x)$ denote a basis for S_r. For $x \in X$ we use the notation

$$\hat{x} \equiv (g_1(x), g_2(x), \ldots, g_s(x));$$

that is, $\hat{x}$ is an s-tuple whose ith coordinate is $g_i(x)$.

For the particular r under consideration define

$$\begin{aligned}
X_{+1} &= \{x \in X: W[x, f(x) - r(x)] = M[f-r]\}\\
X_{-1} &= \{x \in X: W[x, f(x) - r(x)] = -M[f-r]\}\\
X_{+2} &= \{x \in X: f(x) - r(x) = u(x)\}\\
X_{-2} &= \{x \in X: f(x) - r(x) = l(x)\}.
\end{aligned}$$

These are sets of "critical" points. We shall use the notation given above in stating and proving the characterization theorem. First, however, we shall discuss certain exceptional cases which are not of general interest.

Note first that if $X_{+1} \cap X_{-1} \neq \varnothing$ then $M[f-r] = 0$, and hence r must be a solution to (3). Next consider Fig. 1. Here we have drawn a graph of $f(x) - r(x)$

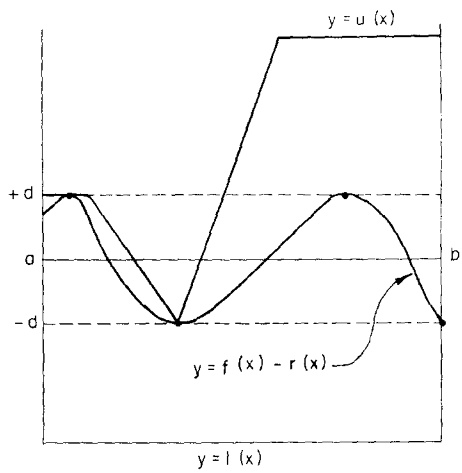

FIG. 1.

when $W(x, y)$ is of the form (4). The particular weighted error curve $W[x, f(x) - r(x)]$ under consideration has four critical points. Considering them from left to right, the first is in both X_{+1} and X_{+2}, the second is in both X_{-1} and X_{+2}, the third is in X_{+1} and the fourth is in X_{-1}. The second point characterizes the solution in the sense that if the unweighted error $f(x) - r(x)$ is either increased or decreased at this point then the absolute value of the weighted error is increased at this point. We summarize the above observations.

LEMMA 1. *If* $(X_{+1} \cup X_{+2}) \cap (X_{-1} \cup X_{-2}) \neq \varnothing$ *then r is a solution to* (3).

THEOREM 1. *Suppose* $M[f - r] < \infty$ *and* $(X_{+1} \cup X_{+2}) \cap (X_{-1} \cup X_{-2}) = \varnothing$. *For this r define*

$$\sigma_r(x) = +1 \text{ if } x \in X_{+1} \cup X_{+2}$$

$$\sigma_r(x) = -1 \text{ if } x \in X_{-1} \cup X_{-2}$$

$$X_r = X_{+1} \cup X_{-1} \cup X_{+2} \cup X_{-2}$$

$$H_r = \text{the convex hull of } \{\sigma_r(x)\,\hat{x} : x \in X_r\}.$$

If r is not a solution to (3) *then one of the following holds*

(a) $0 \notin H_r$.

(b) $0 \in \text{Convex hull of } \{\sigma_r(x)\,\hat{x} : x \in (X_{+2} \cup X_{-2}) \sim (X_{+1} \cup X_{-1})\}$.

Here 0 *is the origin in Euclidean s-space while* $\sim$ *denotes set subtraction.*

Proof. Suppose that $r \equiv p/q$ is not a solution to (3). Then there exists an $r^* \equiv p^*/q^*$ such that $r^* \in R$ and

$$M[f - r^*] < M[f - r]. \tag{9}$$

Observe that if $t \in X_{+1}$ then

$$W[t, f(t) - r(t)] > W[t, f(t) - r^*(t)], \tag{10}$$

and also $W[t, f(t) - r(t)] > 0$. Similarly, if $t \in X_{-1}$ then

$$W[t, f(t) - r(t)] < W[t, f(t) - r^*(t)] \tag{11}$$

and $W[t, f(t) - r(t)] < 0$. Thus if $t \in X_{+1} \cup X_{-1}$ then each of the following is a consequence of (2) and the previous definitions.

$$\begin{aligned}
\sigma_r(t)\, W[t, f(t) - r(t)] &> \sigma_r(t)\, W[t, f(t) - r^*(t)] \\
\sigma_r(t)\, [f(t) - r(t)] &> \sigma_r(t)\, [f(t) - r^*(t)] \\
\sigma_r(t)\, [r^*(t) - r(t)] &> 0 \\
\sigma_r(t)\, [p^*(t) - q^*(t)\, r(t)] &> 0.
\end{aligned} \tag{12}$$

Next observe that if $t \in X_{+2}$ then using (2), and noting that $\sigma_r(t) = +1$,

$$W[t, f(t) - r(t)] \geqslant W[t, f(t) - r^*(t)]$$

$$f(t) - r(t) \geqslant f(t) - r^*(t)$$

$$\sigma_r(t)\,[r^*(t) - r(t)] \geqslant 0 \tag{13}$$

$$\sigma_r(t)\,[p^*(t) - q^*(t)\,r(t)] \geqslant 0.$$

Finally, if $t \in X_{-2}$ then $\sigma_r(t) = -1$,

$$W[t, f(t) - r(t)] \leqslant W[t, f(t) - r^*(t)],$$

and the inequality (13) follows.

To summarize our progress so far, we have proved that if r is not a solution to (3) then the function $h \equiv p^* - q^* r$ is a point of S_r satisfying

$$\begin{aligned} &\text{(a)}\ \ \sigma_r(x)\,h(x) > 0\ \forall\ x \in X_{+1} \cup X_{-1} \\ &\text{(b)}\ \ \sigma_r(x)\,h(x) \geqslant 0\ \forall\ x \in X_{+2} \cup X_{-2}. \end{aligned} \tag{14}$$

If $0 \in H_r$ then there exist points $x_1, x_2, \ldots, x_k \in X_r$ and constants $a_1, a_2, \ldots, a_k$, where $k \leqslant s + 1$, such that

$$a_i > 0 \qquad i = 1, \ldots, k$$

$$\sum_{i=1}^{k} a_i = 1$$

$$0 = \sum_{i=1}^{k} a_i\,\sigma_r(x_i)\,\hat{x}_i.$$

Using our basis for S_r, the last equality becomes

$$0 = \sum_{i=1}^{k} a_i\,\sigma_r(x_i)\,g_j(x_i) \qquad j = 1, \ldots, s. \tag{15}$$

Suppose that

$$h(x) \equiv \sum_{i=1}^{s} b_i\,g_i(x). \tag{16}$$

Then

$$\sum_{i=1}^{k} a_i\,\sigma_r(x_i)\,h(x_i) \geqslant 0$$

with strict inequality if any of the points x_i is in $X_{+1} \cup X_{-1}$. However, using (15) and (16)

$$\sum_{i=1}^{k} a_i \sigma_r(x_i) h(x_i) = \sum_{i=1}^{k} a_i \sigma_r(x_i) \sum_{j=1}^{s} b_j g_j(x_i)$$

$$= \sum_{j=1}^{s} b_j \sum_{i=1}^{k} a_i \sigma_r(x_i) g_j(x_i) = 0.$$

Thus we conclude that no x_i is in $X_{+1} \cup X_{-1}$. This completes the proof of Theorem 1.

At this point we recall the following definition. If H is a finite dimensional subspace of $C(X)$ of dimension k, it is called a Haar subspace if every nonzero element of H has at most $k-1$ zeros.

The following lemma yields immediately a corollary to Theorem 1.

LEMMA 2. *Let H be a Haar subspace of $C(X)$ with a basis $h_1(x), \ldots, h_n(x)$. Let Y be a closed subset of X and let $\sigma_r(y)$ be a continuous nonvanishing real-valued function on Y. Then*

$$0 \in \textit{Convex hull}\, \{\sigma_r(y)\,[h_1(y), h_2(y), \ldots, h_n(y)] \colon y \in Y\} \tag{17}$$

iff $h \in H$ and $\sigma_r(y)\,h(y) \geqslant 0$ for all $y \in Y$ imply $h \equiv 0$.

Proof. Assume (17) holds. If there is a nonzero $h \in H$ such that $\sigma_r(y) h(y) \geqslant 0\ \forall\, y \in Y$, then $\sigma_r h$ has at most $k < n$ zeros $x_1, \ldots, x_k$. Thus there exists an $h_0 \in H$ such that $\sigma_r(x_i) h_0(x_i) = 1$ for $i = 1, \ldots, k$. Consequently, for sufficiently small $\lambda > 0$, $\sigma_r(h + \lambda h_0)$ is strictly positive on Y. This contradiction completes the first half of the proof. The remaining part of the lemma is a standard result [2].

COROLLARY 1. *Suppose that the hypotheses of Theorem 1 are satisfied, and that S_r is a Haar subspace of $C(X)$. Then $0 \notin H_r$.*

THEOREM 2. *Suppose that $M[f - r] = d < \infty$ and r is a solution to (3). Define*

$$S_1 = \{\hat{x} \colon x \in X_{+1} \cup X_{+2}\}$$

$$S_2 = \{-\hat{x} \colon x \in X_{-1} \cup X_{-2}\}$$

$$H_r = \textit{Convex hull of } S_1 \cup S_2.$$

Then the origin of Euclidean s-space lies in H_r.

Proof. We first consider two uninteresting cases. If $M[f-r]=0$ then each point of X is in both X_{+1} and X_{-1}. Thus if $t \in X$ is arbitrary, $\hat{t} \in S_1$ and $-\hat{t} \in S_2$. Since $0=(\frac{1}{2})[\hat{t}-\hat{t}]$ the result is proved.

The second uninteresting case occurs when $M[f-r]>0$ and

$$(X_{+1} \cap X_{-2}) \cup (X_{-1} \cap X_{+2}) \neq \varnothing .$$

In this case also there exists a $t \in X$ such that $\hat{t} \in S_1$ and $-\hat{t} \in S_2$.

For the remainder of the proof we assume $0 < M[f-r] = d < \infty$ and

$$(X_{+1} \cap X_{-2}) \cup (X_{-1} \cap X_{+2}) = \varnothing . \tag{18}$$

With these hypotheses the following notation is well defined.

$$\sigma_r(x) = \begin{cases} +1 & x \in X_{+1} \cup X_{+2} \\ -1 & x \in X_{-1} \cup X_{-2}. \end{cases}$$

The convex hull H_r under consideration is the convex hull of $\{\sigma_r(x)\hat{x} : x \in X_r\}$.

Suppose that the theorem is false. Then by a classical result [*2*] there exists an $h \in S_r$, where

$$h \equiv -p^* + rq^*$$

such that

$$\sigma_r(x)\,h(x) > 0 \;\forall\; x \in X_r. \tag{19}$$

Setting $r \equiv p/q$, define

$$r_\lambda \equiv \frac{p-\lambda p^*}{q-\lambda q^*}.$$

Observe that

$$r - r_\lambda = \frac{-\lambda h}{q-\lambda q^*}. \tag{20}$$

Because $q(x) > 0 \;\forall\; x \in X$, $\exists\; \lambda_1 > 0$ such that if $0 \leqslant \lambda \leqslant \lambda_1$ then $q(x) - \lambda q^*(x) > 0 \;\forall\; x \in X$. We shall restrict our arguments to such λ.

Observe that

$$\begin{aligned} f(x) - r_\lambda(x) &= f(x) - r(x) + [r(x) - r_\lambda(x)] \\ &= f(x) - r(x) - \frac{\lambda h(x)}{q(x) - \lambda q^*(x)}. \end{aligned} \tag{21}$$

We shall show that there exists a constant $\lambda_6 > 0$ such that if $0 < \lambda \leqslant \lambda_6$ then

$$M[f - r_\lambda] < M[f - r]. \tag{22}$$

To begin the argument note that X_{+2} and X_{-2} are compact; hence, using (18) it follows that there exists a $d_1 > 0$, where $d_1 < d$, such that

$$W[x, f(x) - r(x)] \geqslant -d_1 \ \forall\ x \in X_{+2}$$
$$W[x, f(x) - r(x)] \leqslant d_1 \ \forall\ x \in X_{-2}.$$

Note also that $x \in X_{+2}$ imples $f(x) - r(x) > l(x)$ and $x \in X_{-2}$ implies $f(x) - r(x) < u(x)$.

We now examine the set X_{+2}. Let $t \in X_{+2}$ be arbitrary. Then there exists a number $\mu(t)$ satisfying $0 < \mu(t) \leqslant \lambda_1$ such that if $0 \leqslant \lambda \leqslant \mu(t)$ then the following hold.

$$W[t, f(t) - r_\lambda(t)] \geqslant -\left(\frac{d_1 + d}{2}\right)$$
$$f(t) - r_\lambda(t) \geqslant \frac{u(t) + l(t)}{2}.$$

Note that $h(x)$ is continuous, and there exists a neighborhood of t in which $\sigma_r h > 0$. Consequently by continuity arguments we conclude that there exists an open neighborhood $N(t)$ about t such that $x \in N(t)$ and $0 \leqslant \lambda \leqslant \mu(t)$ imply

$$\begin{aligned} W[x, f(x) - r_\lambda(x)] &> -\left(\frac{3d + d_1}{4}\right) \\ u(x) \geqslant f(x) - r_\lambda(x) &> l(x). \end{aligned} \tag{23}$$

The sets $N(x)$, $x \in X_{+2}$, form an open cover of X_{+2}. Suppose $N(x_1), \ldots, N(x_k)$ is a finite subcover. Let

$$\lambda_2 = \min_{1 \leqslant i \leqslant k} \mu(x_i)$$
$$Z_1 = \bigcup_{1 \leqslant i \leqslant k} N(x_i).$$

Then if $x \in Z_1$ and $0 \leqslant \lambda \leqslant \lambda_2$, (23) holds.

Using a similar argument for the points of X_{-2} we conclude that there exist an open set Z_2 containing X_{-2}, and a λ_3 satisfying $0 < \lambda_3 \leqslant \lambda_2$, such that if $x \in Z_2$ and $0 \leqslant \lambda \leqslant \lambda_3$ then

$$\begin{aligned} W[x, f(x) - r_\lambda(x)] &< \frac{3d + d_1}{4} \\ l(x) \leqslant f(x) - r_\lambda(x) &< u(x). \end{aligned} \tag{24}$$

Let $Y \equiv X \sim (Z_1 \cup Z_2)$. Then Y is compact and contains no points of X_{+2} or X_{-2}. Consequently there exists a constant $c > 0$ such that $x \in Y$ implies

$$l(x) + c \leqslant f(x) - r(x) \leqslant u(x) - c. \tag{25}$$

Thus there exists a λ_4 satisfying $0 < \lambda_4 \leqslant \lambda_3$ such that $0 \leqslant \lambda \leqslant \lambda_4$ and $x \in Y$ imply

$$l(x) < f(x) - r_\lambda(x) < u(x). \tag{26}$$

For notational convenience define

$$e(x) \equiv f(x) - r(x)$$
$$s(x) \equiv \operatorname{sgn} e(x).$$

Using the compactness of $X_{+1} \cup X_{-1}$ and (19) it follows that

$$\delta \equiv \left[\min_{x \in X_{+1} \cup X_{-1}} s(x) h(x) \right] > 0.$$

Now define

$$Z_3 = \{x \in X: |W[x, e(x)]| > d/2 \text{ and } s(x) h(x) > \delta/2\}. \tag{27}$$

Observe that Z_3 is open. Thus $X \sim Z_3$ is compact and disjoint from $X_{+1} \cup X_{-1}$. Consequently there exists a constant $c_1 > 0$ such that for all $x \in X \sim Z_3$,

$$|W[x, e(x)]| \leqslant c_1 < d.$$

By a standard continuity argument there exists a λ_5 satisfying $0 < \lambda_5 \leqslant \lambda_4$ such that $0 \leqslant \lambda \leqslant \lambda_5$ and $x \in X \sim Z_3$ imply

$$|W[x, f(x) - r_\lambda(x)]| \leqslant \frac{c_1 + d}{2}. \tag{28}$$

Let Z_4 be the closure of Z_3. Then for $x \in Z_4$

$$|W[x, e(x)]| \geqslant d/2, \qquad s(x) h(x) \geqslant \frac{\delta}{2}.$$

Let

$$\mu = \inf \{|y| : |W[x, y]| \geqslant d/2 \text{ for some } x \in Z_4\}.$$

Then $\mu > 0$ by the compactness of Z_4 and the properties of $W(x, y)$. Finally, select λ_6 such that $0 < \lambda_6 \leqslant \lambda_5$, and $0 \leqslant \lambda \leqslant \lambda_6$ implies $\|r - r_\lambda\| < \mu$. Then if $x \in Z_4$ and $0 \leqslant \lambda \leqslant \lambda_6$ we have

$$\operatorname{sgn} [f(x) - r_\lambda(x)] = \operatorname{sgn} [f(x) - r(x)].$$

Moreover, if $x \in Z_4$ and $0 < \lambda \leqslant \lambda_6$ then

$$|W[x, f(x) - r_\lambda(x)]| < d. \tag{29}$$

The inequalities (28) and (29) taken together imply that if $0 < \lambda \leqslant \lambda_6$ then (22) is satisfied.

COROLLARY 2. *Let $f \in C(X)$ and suppose $r \in R$ is a solution to* (3). *If S_r is a Haar subspace and $(X_{+1} \cap X_{-2}) \cup (X_{-1} \cap X_{+2}) = \varnothing$ then r is unique.*

Proof. If $f \in R$ then $r \equiv f$. Thus if r_0 is any solution to (3) we must have $r_0 \equiv r$.

Suppose then that $f \notin R$ and r_0, r are both solutions to (3). Then for all $x \in X_r$

$$\sigma_r(x)[f(x) - r(x)] \geqslant \sigma_r(x)[f(x) - r_0(x)].$$

This implies

$$\sigma_r(x)[r_0(x) - r(x)] \geqslant 0.$$

If $r_0 \equiv p_0/q_0$ where $q_0 > 0$ then for all $x \in X_r$

$$\sigma_r(x)[p_0(x) - q_0(x)r(x)] \geqslant 0.$$

Using Lemma 2 we conclude $p_0 - q_0 r \equiv 0$; so $r_0 \equiv r$.

THEOREM 3. *Let X be an interval $[a,b]$ and let r be such that $\infty > M[f - r] > 0$. If S_r is a Haar subspace and $(X_{+1} \cap X_{+2}) \cup (X_{-1} \cap X_{-2}) = \varnothing$, then r is a solution to* (3) *iff there exist points $x_1 < x_2 < \ldots < x_{s+1}$ in X_r such that*

$$\sigma_r(x_i) = (-1)^{i+1}\sigma_r(x_1) \qquad i = 2, \ldots, s+1.$$

Here s is the dimension of S_r.

Proof. By Corollary 1 and Theorem 2, r is a solution iff $0 \in H_r$. By a standard argument ([*1*], page 74) $0 \in H_r$ iff there exist $s+1$ points $x_1 < x_2 < \ldots < x_{s+1}$ in X_r such that $\sigma_r(x_i) = (-1)^{i+1}\sigma_r(x_1)$, $i = 2, \ldots, s+1$.

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